Synchronization of hyperchaotic systems with delayed bidirectional coupling

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Synchronization of three bidirectionally coupled hyperchaotic systems is studied, using the Ikeda model as the hyperchaotic unit. The whole system is given by three delay-differential equations with two distinct time lags. Sufficient condition for the global stability of the manifold of exact synchronization is found analytically. Local stability of the synchronization manifold is studied by numerical computation of the transverse Lyapunov exponent and statistical properties of the orbits.

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I. INTRODUCTION

Synchronization of chaotic systems is most often studied for systems with relatively low dimensional chaotic attractor, for example using the models of Rossler or Lorenz (an overview and an extensive list of references can be found in Ref. [1]). It is of considerable interest to study the synchronization between hyperchaotic systems, i.e., the systems with a chaotic attractor, such that there are at least two positive Lyapunov exponents for the restriction of the system on the attractor. Examples of such systems, for which the problem of synchronization has been analyzed, are given by electronic circuits and networks of such elements (for example, Refs. [2,3]). Another class of hyperchaotic systems, which will be considered here, is provided by the semiflows generated by delay-differential equations (DDEs). As is well known, a simple nonlinear scalar DDE with a single fixed time lag τ, $\dot{x}(t)=f(x(t),x(t-\tau))$, gives an infinite dimensional dynamical system on the phase space $C(-\tau,0)$ of continuous functions on the interval $(-\tau,0)$ [4]. Large τ usually implies high-dimensional chaotic attractor [5–7].

Besides the theoretical interest, as examples of hyperchaotic systems, the models given by DDEs often appear in applications, for example in biology, nonlinear optics, or secure communication. Furthermore, such potentially hyperchaotic units could appear as constitutive elements of complex systems, and can transmit excitations between them. In biological, as well as physical, applications the transmission of excitations is certainly not instantaneous, and the representation by nonlocal and instantaneous interactions should be considered only as a very crude approximation. An important physical example is given by coupled lasers in a chaotic state with electro-optical or optical feedback [8,9]. Thus, it is of some interest to study the collective behavior of systems composed of several chaotic units which are coupled by time-delayed interaction, and such that each unit if decoupled from the system would have a hyperchaotic attractor due to an intrinsic time lag.

Delayed coupled regular oscillators have been extensively studied, for example, in relation to realistic neuronal networks with synaptic delays [10]. Also, excitable systems with delayed coupling have been recently analyzed [11]. Synchronization of two instantaneously coupled time-delayed hyperchaotic systems in a master-slave configuration has been studied in Ref. [12]. Later, Shahverdiev *et al.* [13,14] applied the same methods to the same system but with the delayed coupling, and studied lag and anticipating synchronization. We are not aware of any study of the hyperchaotic systems with bidirectional delayed coupling.

We shall study the synchronization of three identical Ikeda systems with bidirectional diffusive coupling. As is well known, the dimension of the attractor of a single Ikeda system, $\dot{x}(t)=-x(t)+\mu\sin[x(t-\tau_1)]$, increases with the time lag τ_1. For a sufficiently large τ_1 and μ the system is hyperchaotic. In fact, the number of positive Lyapunov exponents increases with τ_1, but the size of the exponents decreases, so that the dimension of the attractor saturates. The character of chaos, in particular the number of positive Lyapunov exponents and their values, for a single Ikeda equation has been thoroughly studied [6,7,15]. An example of the attractor, for $\mu=3$ and $\tau_1=30$, with at least two positive Lyapunov exponents is illustrated in Fig. 1. The whole system is given by

$$\dot{x}_i=-x_i+\mu\sin(x_i^{\tau_1})+c(x_{i-1}^{\tau_2}+x_{i+1}^{\tau_2}-2x_i),$$
$$i=1,2,3;\quad x_4\equiv x_1, \tag{1}$$

where $x_i^{\tau_j}(t)\equiv x_i(t-\tau_j)$.

The system of DDE (1) has two different time lags τ_1 and τ_2, which makes its analysis more difficult than for the DDE with only one time lag. Furthermore, the two time lags appear in different ways and play quite different roles. The time lag τ_1 is an intrinsic "parameter" of each of the units and controls the complexity of the dynamics of the uncoupled unit. On the other hand, the time lag τ_2 measures the time needed for the transfer of information between the units, and, contrary to τ_2, appears linearly in Eqs. (1). Since we are predominantly interested in the system with highly chaotic units, it would be natural to assume that $\tau_1>\tau_2>0$.

We shall first prove that exact synchronization $x_1=x_2=x_3$ necessary occurs for a sufficiently large coupling c and for any values of the time lags $\tau_1\geqslant\tau_2\geqslant 0$; namely, for any $\tau_1\geqslant\tau_2\geqslant 0$ and c larger than $c_0(\mu)=(\mu-1)/2>0$ (and independent of τ_1,τ_2) the global attractor of the system (3)

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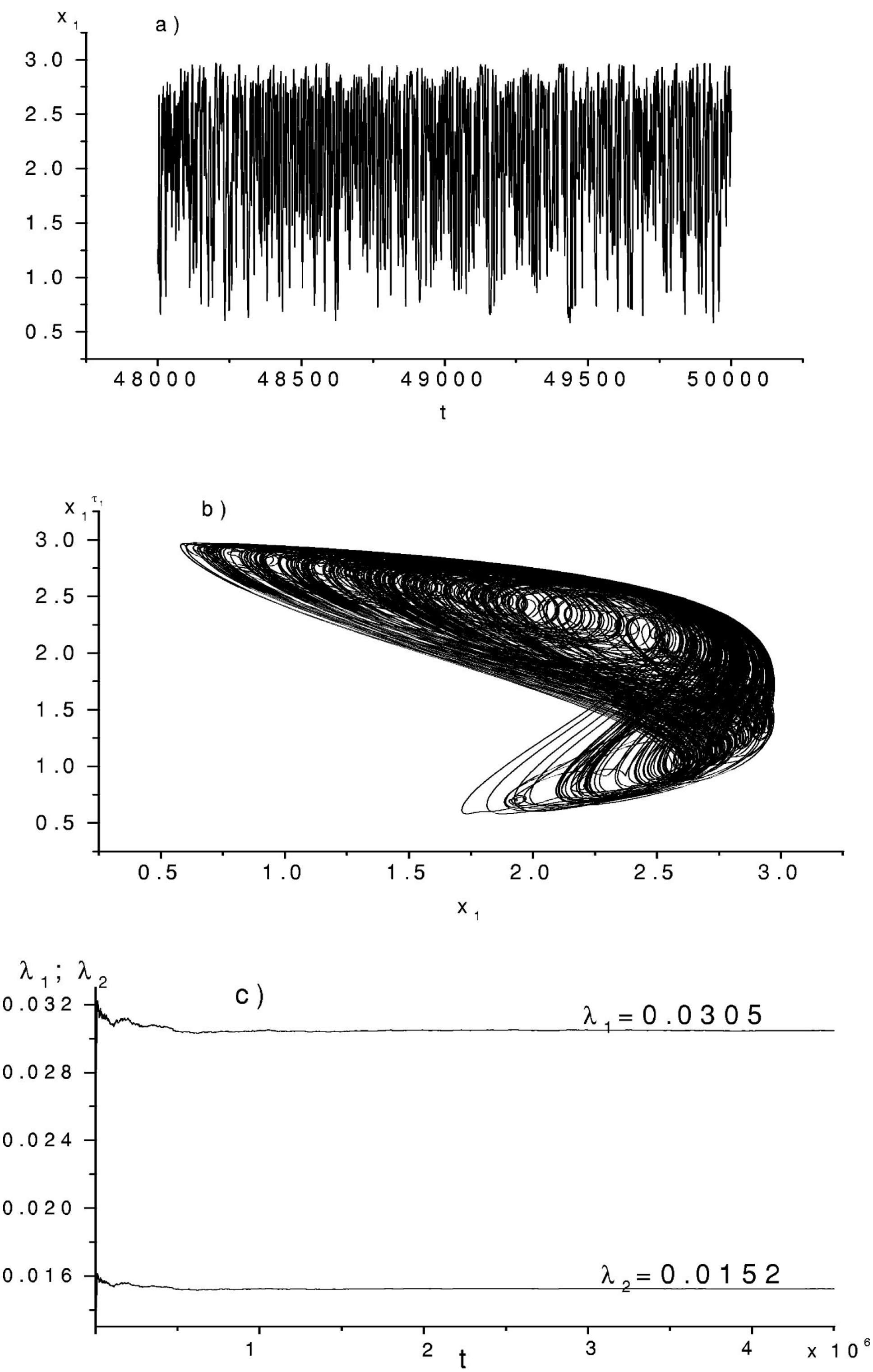

FIG. 1. Illustration of the hyperchaotic attractor of the single Ikeda system ($\mu=3,\tau_1=30$): (a) time series $x_1(t)$, (b) projection on $x_1, x_1^{\tau_1}$, (c) convergence of calculations of two positive Lyapunov exponents.

satisfies $x_1=x_2=x_3$. In the opposite case $\tau_1<\tau_2$ numerical evidence indicates that a sufficiently large c also implies the exact synchronization. Then, we discuss the local attracting properties of the synchronization manifold $x_1=x_2=x_3$. To this end, we compare the information obtained by numerical computation of the statistical correlations along typical orbits and the transverse Lyapunov exponents. It turns out that, also for some $c<c_0$, the manifold of exact synchronization is locally stable but could coexist with other stable low-dimensional attractors corresponding to more general types of synchronization. Naturally, no synchronization occurs for sufficiently small coupling c.

II. GLOBAL STABILITY OF THE EXACTLY SYNCHRONOUS SOLUTIONS

In order to study the exact synchronization, we analyze the dynamics of the following two variables:

$$\Delta_{1,2}=x_1-x_2, \quad \Delta_{2,3}=x_2-x_3. \tag{2}$$

The dynamics of each of these two functions is given by a scalar DDE of the same form

$$\dot{\Delta}=-(1+2c)\Delta-c\Delta^{\tau_2}+\sigma(t)\sin(\Delta^{\tau_1}/2), \tag{3}$$

where $\Delta=\Delta_{1,2},\Delta^{\tau_1}=\Delta_{1,2}(t-\tau_1),\Delta^{\tau_2}=\Delta_{1,2}(t-\tau_2)$ or $\Delta=\Delta_{2,3},\Delta^{\tau_1}=\Delta_{2,3}(t-\tau_1),\Delta^{\tau_2}=\Delta_{2,3}(t-\tau_2)$. In the two cases the time dependent parameter $\sigma(t)$ is given by

$$\sigma(t)=2\mu\cos\frac{x_1^{\tau_1}+x_2^{\tau_1}}{2} \quad \text{or} \quad \sigma(t)=2\mu\cos\frac{x_2^{\tau_1}+x_3^{\tau_1}}{2}. \tag{4}$$

Although, the time dependence of $\sigma(t)$ could be quite complicated its absolute value is always bounded by 2μ.

The trivial stationary solution of the scalar DDE (3), $\Delta(t)=0$, corresponds to the exactly synchronous solution $x_1(t)=x_2(t)=x_3(t)$ of Eq. (1). Global asymptotic stability of $\Delta(t)=0$ implies that the global attractor of Eq. (1) satisfies $x_1=x_2=x_3$. To find a sufficient condition for the global asymptotic stability of $\Delta=0$ it is useful (see for example Refs. [12,16]) to consider Eq. (3) as a dynamical system on the phase space given by continuous functions Δ defined on the interval $[-\tau,0]$, where $\tau=\max\{\tau_1,\tau_2\}$, with the norm $||\Delta||^2=\int_{-\tau}^{0}\Delta^2(\theta)d\theta$. Solutions of Eq. (3) for different initial functions in $C[-\tau,0]$ generate a semiflow on this phase space given by $\Delta_t(\theta)=\Delta(t-\theta),t\in R^+,\theta\in[-\tau,0]$. Thus the norm of an initial function Δ_0 evolves according to

$$||\Delta_t||^2=\int_{-\tau}^{0}\Delta_t^2(\theta)d\theta=\int_{-\tau}^{0}\Delta(t-\theta)d\theta. \tag{5}$$

The functional which we analyze is related to the norm (5) and is given by

$$L(\Delta_t)=\frac{\Delta(t)^2}{2}+b\int_{-\tau}^{0}\Delta^2(t-\theta)d\theta, \tag{6}$$

where b is a parameter to be determined. Formula (6) defines a Lyapunov functional for Eq. (3) if there is a value of the parameter b such that Eq. (6) is a decreasing function of t along the solutions of Eq. (3). In this case the stationary solution $\Delta=0$ is globally asymptotically stable. The derivative of L along the solution of Eq. (3) is given by

$$\dot{L}(t)=-(1+2c)\Delta^2-c\Delta\Delta^{\tau_2}+\sigma(t)\Delta\sin(\Delta^{\tau_1}/2)+b\Delta^2 -b\Delta^{\tau}\equiv V(\Delta,\Delta^{\tau_1},\Delta^{\tau_2}). \tag{7}$$

The properties of the functional L depend on the relation between τ_1 and τ_2 via the lower boundary $\tau=\max\{\tau_1,\tau_2\}$ of the interval on which the initial functions must be given. This is reflected in the different properties of the functions $V(\Delta,\Delta^{\tau_1},\Delta^{\tau_2})$ in the two cases. In the case $\tau_1<\tau_2$ the function $V(\Delta,\Delta^{\tau_1},\Delta^{\tau_2})$ has a single extremal point at $(\Delta,\Delta^{\tau_1},\Delta^{\tau_2})=(0,0,0)$ where $V(0,0,0)=0$. This extremum is necessarily a maximum if $b=\mu/2$ and for any $c>(\mu-1)/2\equiv c_0$. Thus, if $c>c_0$ the functional (6) where $b=\mu/2$ is the Lyapunov functional for Eq. (3) that proves the global asymptotic stability of the zero stationary solution of Eq. (3). This means that the dynamics on the attractor of Eq. (1) with $\tau_1>\tau_2$ is always synchronous provided that $c>c_0$.

In the complementary case $\tau_2>\tau_1$ the corresponding function $V(\Delta,\Delta^{\tau_1},\Delta^{\tau_2})$ has several local extrema and nothing can be concluded about the global stability of $\Delta=0$ using the functional (6). However, numerical evidence indicates that also in this case if $c>c_0$ then $\Delta=0$ is the global attractor for Eq. (3).

III. LOCAL STABILITY OF SYNCHRONOUS SOLUTIONS

Numerical calculations show that the the synchronous solutions $x_1=x_2=x_3$ could be locally stable for the values of the coupling constant c smaller than c_0 which implies global stability. The dynamics on this locally stable synchronization manifold could be low-dimensional chaotic, quasiperiodic, or periodic. Furthermore, for $c<c_0$ there could be several coexisting local low-dimensional attractors that describe various types of generalized synchronization. We shall concentrate on the exact synchronization and the other types shall be just briefly mentioned.

In order to describe quantitatively the local stability of the synchronous dynamics and the degree of synchronization for $c<c_0$ we have numerically computed the largest transverse Lyapunov exponent [12] and the lag function [17]. The first quantity is defined using again the norm (5) in analogy with the Lyapunov exponent of finite dimensional systems. For the manifold $x_1=x_2$,

$$\lambda_{1,2}=\lim_{t->\infty}\frac{1}{t}\frac{||\Delta_{1,2;t}||}{||\Delta_{1,2;0}||}, \tag{8}$$

where the norm $||\Delta_{1,2;0}||$ is small, and analogously for $x_2=x_3$. The synchronization manifold is locally attracting for some (c,μ,τ_1,τ_2) if both $\lambda_{1,2}$ and $\lambda_{2,3}$ are negative for at least some sufficiently small $\Delta_{1,2;0}$ and $\Delta_{2,3;0}$. For fixed values of μ,τ_1, and τ_2 the value of the coupling c when this happens is called the (local) exact synchronization threshold [12].

The existence of the stable synchronization could also be detected by computation of some statistical property of typical orbits. One such parameter, which we have used, is the rms deviation

$$\kappa_{0;i,j}=\sqrt{\frac{\langle(x_i-x_j)^2\rangle}{\langle x_i^2(t)x_j^2(t)\rangle}}, \tag{9}$$

where $\langle\ \rangle$ denotes the time average. If there is asymptotically stable exact synchronization, then κ_0 vanishes, and for the nonsynchronized states κ_0 is finite. A more general quantity is derived from the lag function:

$$\kappa(\xi)=\sqrt{\frac{\langle x_i(t)-x_j(t-\xi)^2\rangle}{\langle x_i^2(t)x_j^2(t)\rangle}}. \tag{10}$$

Then, an order parameter is defined by

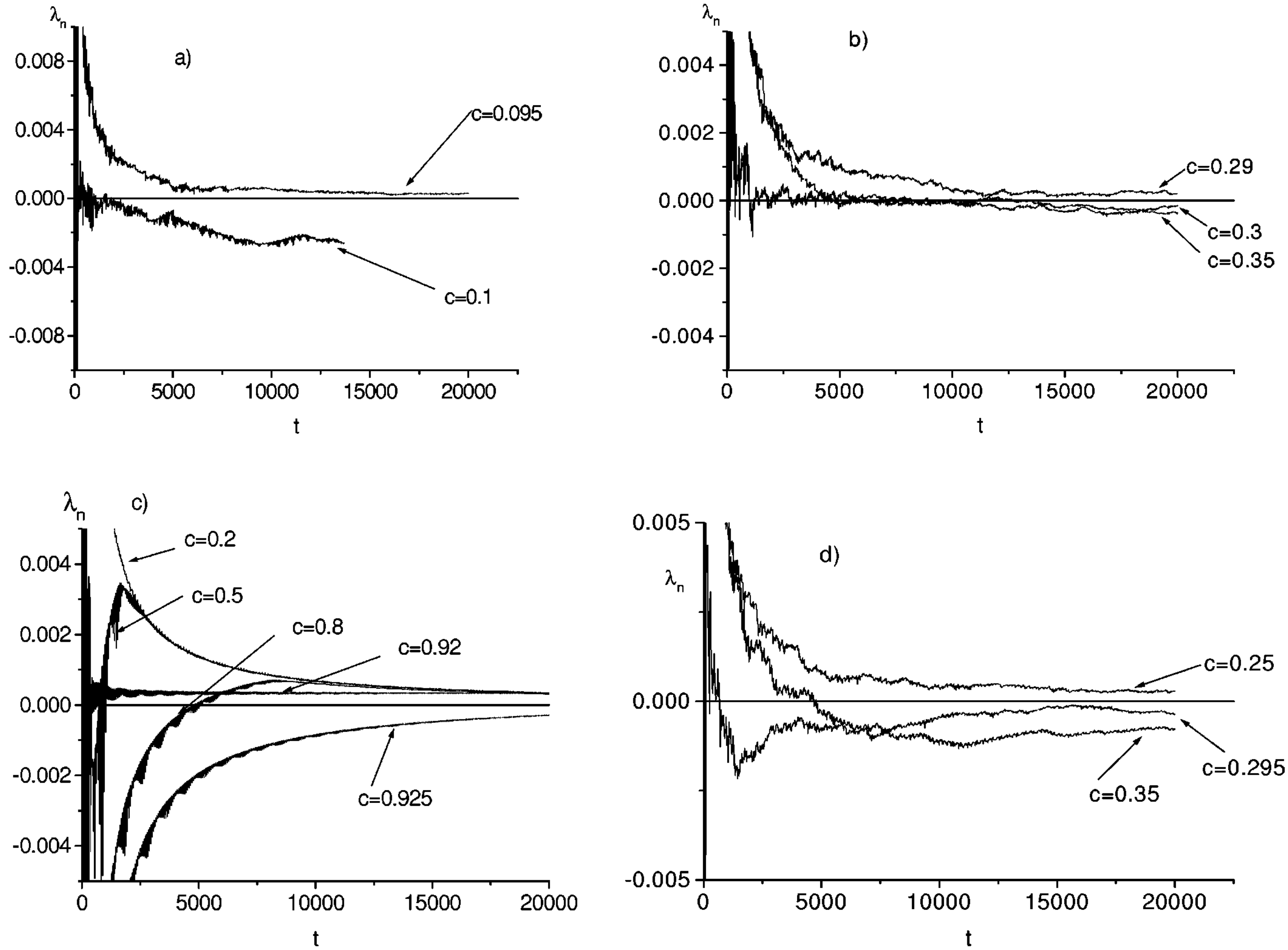

FIG. 2. The transverse Lyapunov exponent λ_n for an orbit starting close to $\Delta_{i,j}=0$. Parameters are $\mu=3,\tau_1=30$ and (a) $\tau_2=0$, (b) $\tau_2=20$, (c) $\tau_2=30$, and (d) $\tau_2=40$.

$$\kappa_{min;i,j}=\min_{\xi}\{\kappa_{ij}(\xi)\}. \tag{11}$$

We shall denote by κ_0 and κ_{min} the sets for all pairs (i,j) of quantities (9) and (11), respectively. The lag function and the order parameter κ_{min} can detect not only the exact synchronization, such as κ_0 but also the synchronization between $x_i(t)$ and $x_j(t-\xi)$ for some time lag ξ. If the dynamics is nonergodic with no global attractor these quantities could depend on the initial functions. However if there is locally stable synchronization manifold, it can be detected in numerical computations by almost vanishing values of κ_{min}, for any sufficiently close initial conditions.

Results of the numerical computation of either the Lyapunov coefficient or the rms deviation and the lag function lead to qualitatively and quantitatively the same conclusions. We fixed the pair of the intrinsic parameters (μ,τ_1) to some typical values and studied the exponents $\lambda_{1,2}$ and $\lambda_{2,3}$ and the statistical quantities as functions of c and τ_2, for initial conditions in a small neighborhood of $\Delta_{1,2;0}=\Delta_{2,3;0}=0$. A sample of our calculations, for $(\mu,\tau_1)=(3,30)$, is illustrated in Figs. 2–6. Computations for other values of (μ,τ_1) such that the single isolated unit is hyperchaotic lead to qualitatively the same conclusions (illustrated in Fig. 7). We can summarize the numerical calculations with the following conclusions. When $\tau_2=0$, the local exact synchronization threshold is $c_{0,l}\approx 0.1$ and is smaller than that for any $\tau_2\neq 0$. In the latter case and for $0\neq\tau_2<\tau_1=30$ the local exact synchronization threshold $c_{0,l}$ is $c_{0,l}\approx 0.3$, and is much smaller than c_0. Locally attracting manifold of exact synchronization could coexist with other locally stable low-dimensional attractors. One such attractor is illustrated in Fig. 4.

For $\tau_1=\tau_2$ the numerical evidence strongly supports the conclusion that the only exactly synchronous state is the stable stationary solution $x_1=x_2=x_3=\text{const}\neq 0$. There are other stable low-dimensional manifolds that correspond to more general types of synchronization [see Figs. 5(a,b)]. Consequently, κ_0 and κ_{min} are different, as is seen in Fig. 3(c). The function $\Delta_{i,j}(t)$ oscillates on such manifold of generalized synchronization, with the transversal Lyapunov coefficients $\lambda_{1,2}$ and $\lambda_{2,3}$ numerically equal to zero [Figs. 5(c,d)]. The dimension of the synchronization manifold drops down to zero as c is increased. The symmetric stationary state becomes locally stable, and apparently also the glo-

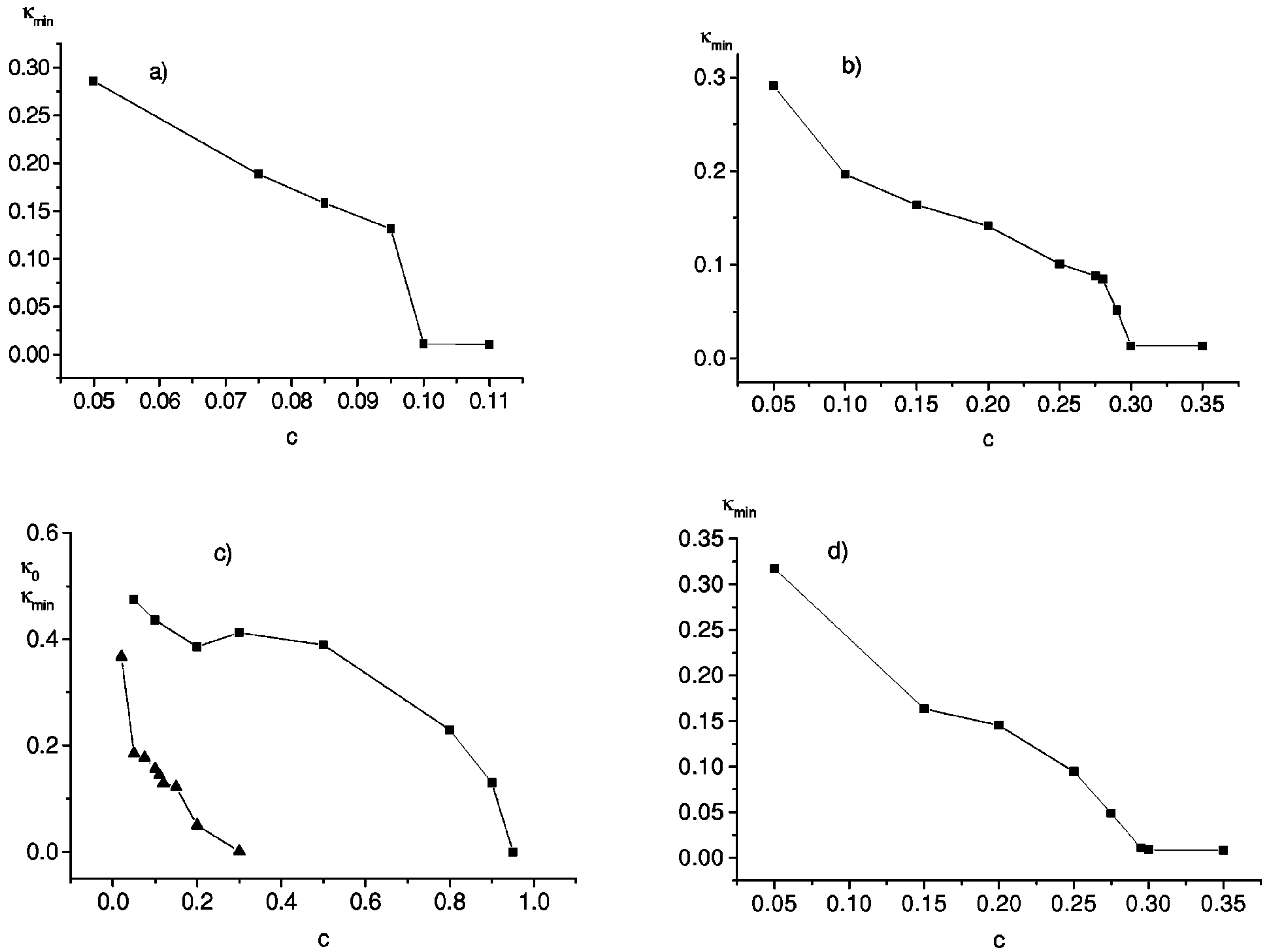

FIG. 3. Order parameter κ_{min} for an orbit starting close to $\Delta_{i,j}=0$. Parameters are $\mu=3,\tau_1=30$ and (a) $\tau_2=0$, (b) $\tau_2=20$, (c) $\tau_2=30$ κ_{min} (triangles), κ_0 (squares), and (d) $\tau_2=40$.

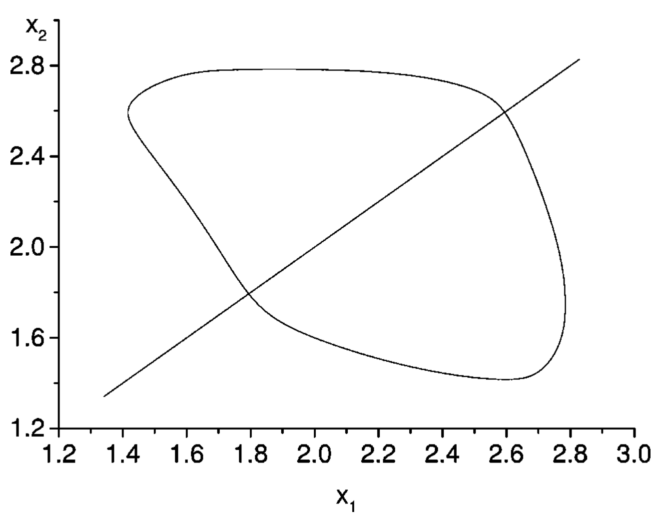

FIG. 4. Projection on (x_1,x_2) of two coexisting low-dimensional locally stable manifolds for $\tau_1=30$, $\tau_2=28$, and $c=0.5$.

bal attractor, for quite large $c\approx0.925$, which is close to the estimated sufficient threshold $c_0=(\mu-1)/2=1$. For larger values of c, numerical computations for various initial functions and different values of $c>c_0$, as large as $c=20$, always gave the symmetric stationary solution as the only attractor. Thus, observed in the direction of decreasing c the synchronization manifold undergoes two successive supercritical Hopf bifurcations, and for c smaller than 0.2 the attractor is multidimensional.

For not very big $\tau_2>\tau_1$ the local exact synchronization is possible. The threshold $c_{0,l}$ is again ≈0.3 and much smaller than $c=c_0$.

Let us now briefly discuss the dynamics on the synchronization manifold. As the coupling c is increased beyond the local synchronization threshold the dynamics could be chaotic, quasiperiodic, periodic and finally the manifold could consist of the single stable stationary state. Various types of dynamics are illustrated in Fig. 6. The numerical evidence suggests that the dynamics on the synchronization manifold is always chaotic for the coupling c larger but close to the local synchronization threshold. Furthermore, we have found values of (μ,τ_1) and the corresponding $c>c_0$ and τ_2 (see

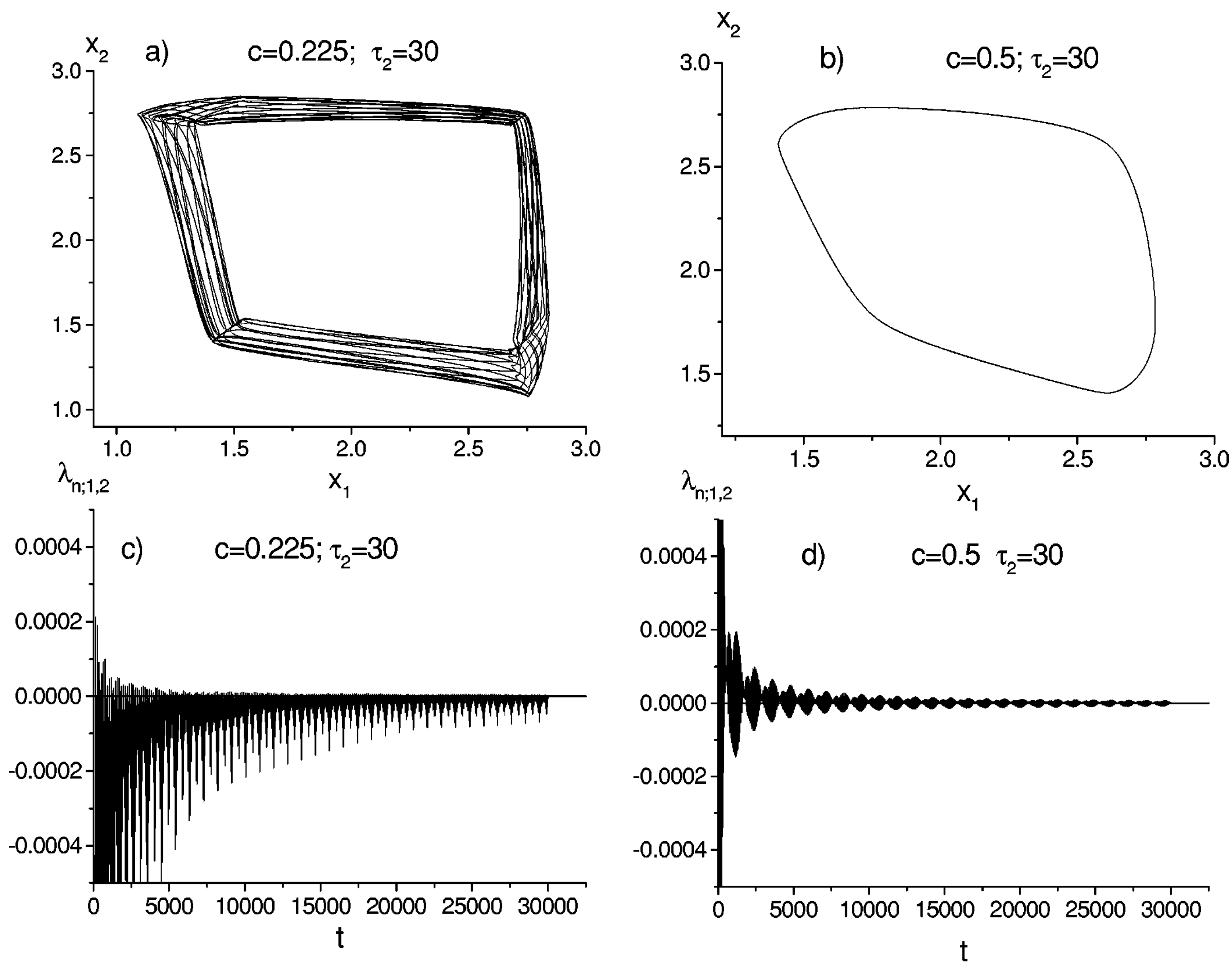

FIG. 5. Generalized synchronization for $\tau_2 = \tau_1$ ($=30$). Projection on (x_1, x_2) of the synchronization manifolds [(a) and (b)] and the transverse Lyapunov coefficients [(c) and (d)] for $c = 0.225$ [(a),(c)] and $c = 0.5$ [(b),(d)].

Fig. 7), such that the dynamics of the single unit is chaotic and the manifold of the exact synchronization is globally stable. Thus, the numerical evidence suggests that the dynamics could be chaotic even when the synchronization manifold is globally stable. As c is increased the dynamics becomes regular. Generally, the sequence of bifurcations as c is increased leads to the stable stationary solution as the synchronization manifold for some $c > c_0$, which depends on τ_2. Even further increase of c destabilizes the stationary solution and leads to a stable limit cycle, still within the synchronization manifold.

We have presented results of the numerical computations for one particular pair of values of the parameters (μ, τ_1). However, qualitatively the same picture was obtained for other values of (μ, τ_1) such that the single unit is hyperchaotic. In particular, in Fig. 7, we illustrate the exact synchronization for $c > c_0$ [Fig. 7(c)], the chaotic dynamics on the synchronization manifold [Figs. 7(a,b)], and the only stable type of generalized synchronization at $\tau_2 = \tau_1, c < c_0$ [Fig. 7(d)] for the system with $(\mu, \tau_1) = (5,50)$.

IV. SUMMARY AND PERSPECTIVES

We have studied synchronization of three identical hyperchaotic systems coupled by diffusive interaction. The hyperchaotic nature of the dynamics of each of the uncoupled units is produced by an intrinsic time delay τ_1, using as the units the Ikeda model, and the time delay τ_2 in the interaction between the units is taken explicitly. Mathematically the system is described by three delay-differential equations with the two different discrete time lags. We have given an analytic estimate of the sufficient value of the coupling constant that implies exact synchronization, in the case when the interaction delay τ_2 is smaller than the intrinsic delay τ_1, which is a plausible assumption in the case of hyperchaotic units. Local stability of the synchronization manifold is studied numerically, using the transverse Lyapunov exponents and the quantities κ_{min} and κ_0, which describe statistical correlations between the component systems. It is shown that, for any relation between τ_1 and τ_2 except for $\tau_1 = \tau_2$, and for moderate values of the coupling constant, there could be more than one locally stable low-dimensional manifolds,

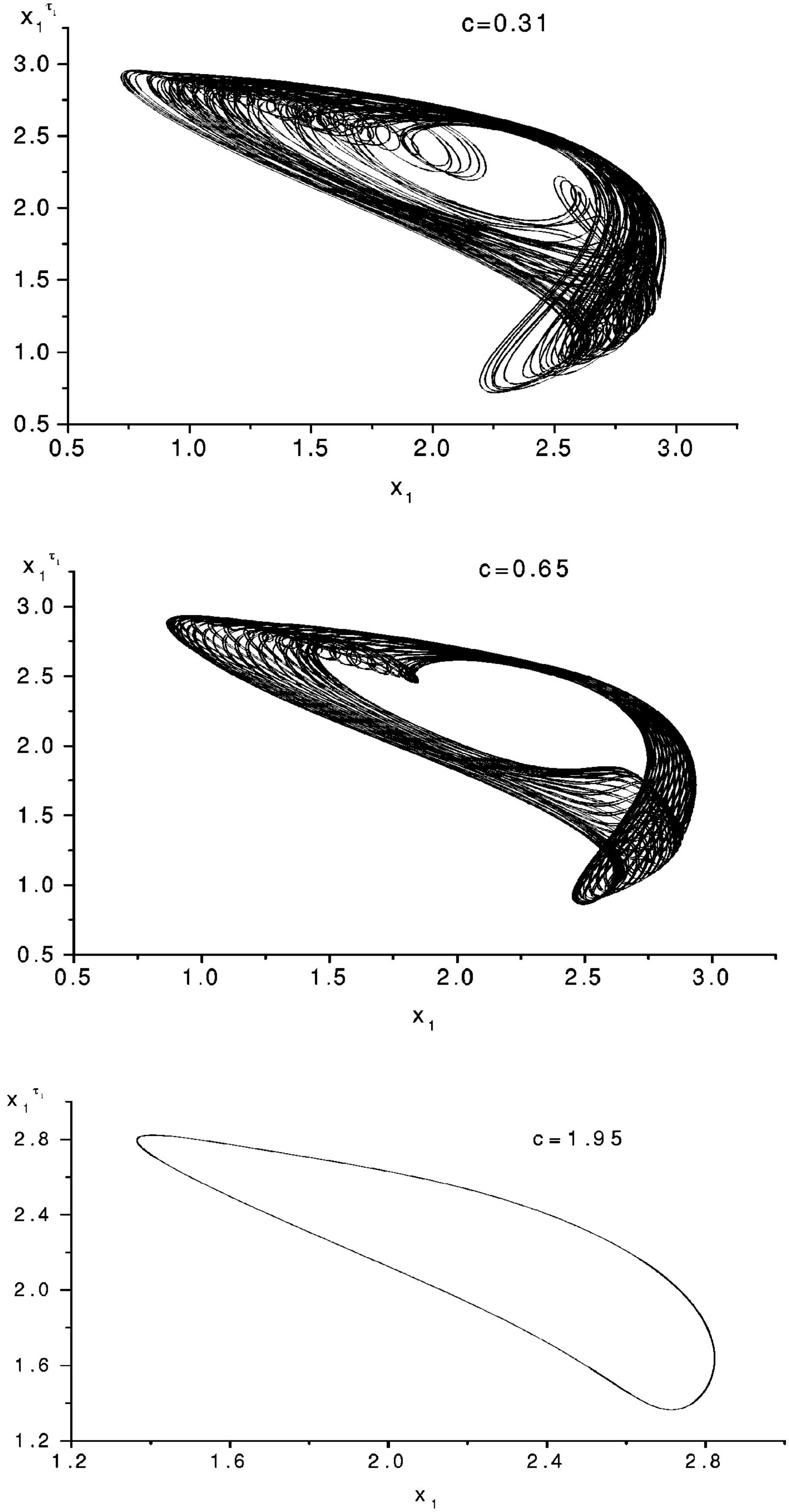

FIG. 6. Illustration of the possible dynamics on the locally stable exact synchronization manifold.

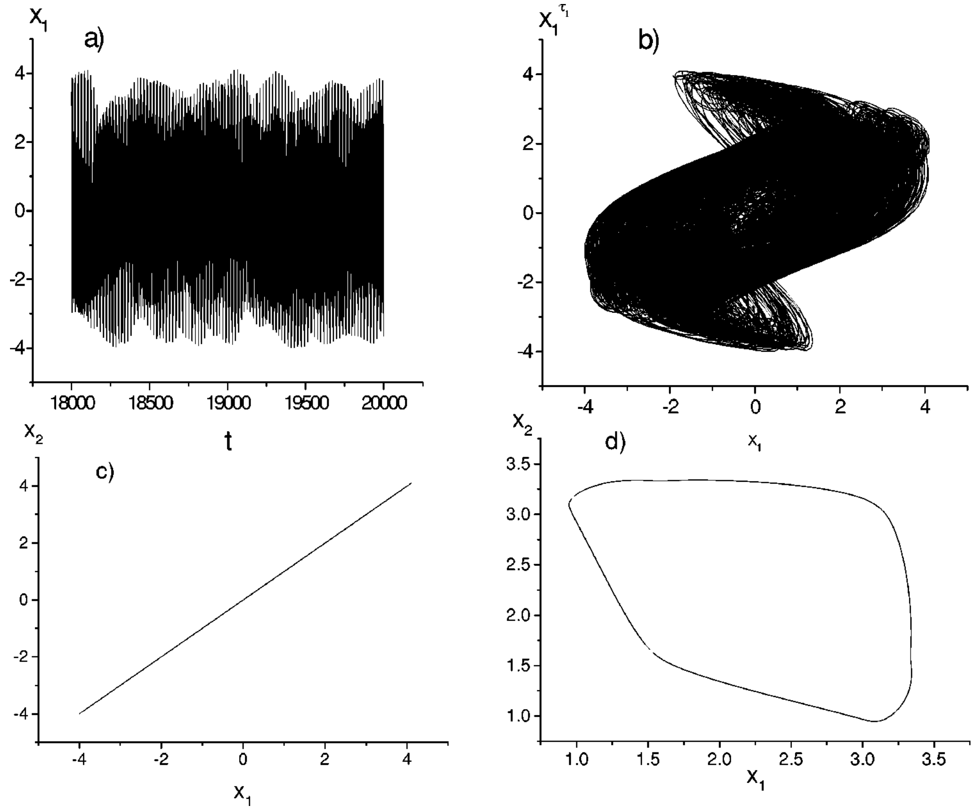

FIG. 7. Illustration of the synchronization for $\mu=5$ and $\tau_1=50$: (a) hyperchaotic time series $x_1(t)$, (b) projection of the attractor on $(x_1,x_1^{\tau_1})$, (c) projection on (x_1,x_2), (d) projection on (x_1,x_2). In (a)–(c) $c=2.01$ and $\tau_2=20$, and in (d) $c=1.5$ and $\tau_2=\tau_1=50$.

which coexist with the manifold of the exact synchronization. The type of synchronization depends on the initial conditions. However, as the coupling constant is increased the only type of synchronization that occurs is the exact synchronization. The dynamics on the manifold of the exact synchronization could be chaotic, even if the manifold is globally attracting.

The work presented in this paper should be extended in several directions. First, in order to test the model dependence, instead of the Ikeda model one could use other delay-differential systems in the hyperchaotic regime. For example, the MacKey-Glass equation apparently leads to similar conclusions. Second, larger chains with more than three units should be analyzed. Much more complex structure of locally stable clusters with synchronized dynamics would become possible. Another line of research would be to study the synchronization of hyperchaotic units with slightly different internal parameters and coupling.

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